GRAPHING RATIONAL FUNCTIONS: GUIDED NOTES (TEACHER GUIDE)

Definitions	Comments
Rational function: $\frac{p(x)}{q(x)} = \frac{a_m x^m + \ldots + a_0}{b_n x^n + \ldots + b_0}$	*Refresher from the "Can't Touch This, Part 1" lesson's Guided Notes.*
Vertical asymptote: a line that a curve approaches and **never** crosses, because we can't divide by zero	*Elaboration of types of asymptotes.*
Horizontal asymptote: a line with a slope of zero that the curve approaches and **sometimes** crosses*	*Elaboration of types of asymptotes.*
Slant (oblique) asymptote: a line with a slope other than zero (and is not undefined) that the curve approaches and **sometimes** crosses*	*Introduction to slant asymptotes.*
**The curve is allowed to cross horizontal and slant asymptotes.*	*This is because those asymptotes give guidance to the ends of the graph. In other words, a horizontal or slant asymptote describes what the graph is doing at its very-far-left end (as x approaches $-\infty$) and at its very-far-right end (as x approaches $+\infty$). Those asymptotes do not describe what the graph is approaching near the center of the graph, like when $x = 0$. Help students see why there is an exception to crossing asymptotes.

Definitions	Comments
• Rational functions can have 0, 1, 2, ... vertical asymptotes. • Rational functions can have 0 or 1 horizontal asymptotes. • Rational functions can have 0 or 1 slant asymptotes.	Some functions have two horizontal asymptotes, but rational functions have a maximum of 1 horizontal asymptote.

How to Graph a Rational Function

Step 1) Find the asymptote(s).

- If the degree on the top is greater than the degree on the bottom, then the ratio for a horizontal asymptote will be a number over zero, which is undefined, so there is **no horizontal asymptote** when $m > n$.
- If the degree on the top is only 1 greater than the degree on the bottom, then you will have a **slant asymptote**.

Step 2) Sketch the asymptote(s) with dashed lines.

- Do not worry about sketching slant asymptotes at this time.

Step 3) Make a table.

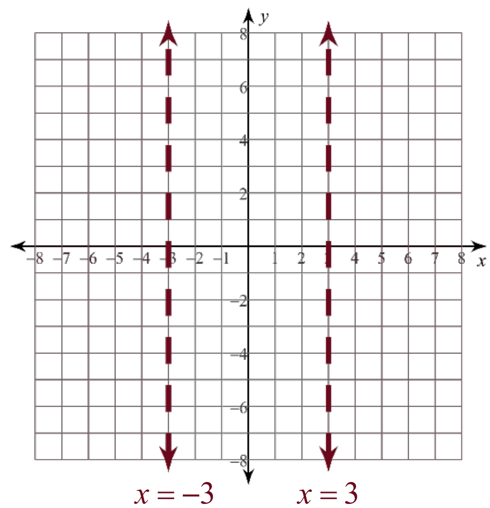
- Pick x-values based on the vertical asymptote(s).
- If there is no vertical asymptote, then let $x = 0$ be the middle number in your table.

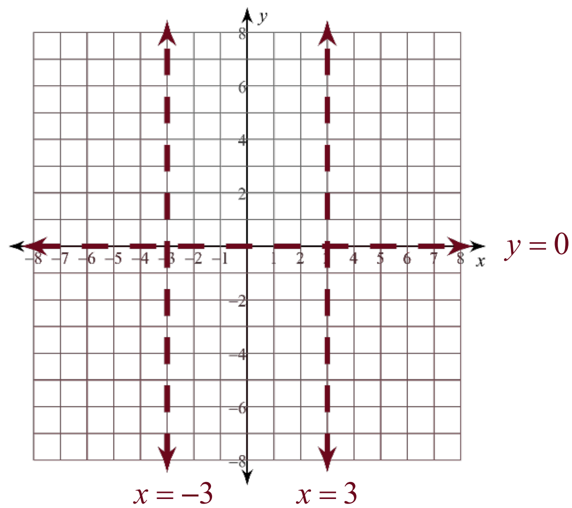
Step 4) Plot points and connect dots.

Steps	Comments
Step 1	*No change in how to find asymptotes from the previous lesson. The addition is the difference between the degrees determines if there is a slant asymptote.*
Step 2	Students often learn how to find the equation for a slant asymptote during a precalculus course, but not in Algebra 2.
Step 3	A vertical asymptote should be in a table with 3 x-values on each side to better see the shape of the graph. When there is no vertical asymptote, then $x = 0$ is often a good number to have in the middle of the table.
Step 4	*No change from previous lesson.*

Graph each function. Be sure to label the asymptote(s).

1) $y = \frac{8x}{x^2 - 9}$

Teacher	Students
Step 1) How do we find the vertical asymptote or asymptotes?	Set the denominator not equal to zero and solve for x. Find out what x can't be. $x^2 - 9 \neq 0$ $x^2 \neq 9$ $x \neq \pm 3$
Why can the denominator not equal zero?	Because we can't divide by zero.
Step 2) If $x \neq \pm 3$, then we have two vertical asymptotes at $x = -3$ and $x = 3$. Let's sketch those asymptotes and label them on our graph.	
Step 1) Now let's find our horizontal asymptote, or at least see if this function has one. How do we find a horizontal asymptote?	Make the degrees on the top and bottom of the fraction be the same.
Specifically, how would we do that?	Since the denominator has the largest degree, rewrite the equation so that both the top and bottom of the fraction have a degree of 2. $y = \frac{0x^2 + 8x}{1x^2 - 9}$

<table>
<tr><th>Teacher</th><th>Students</th></tr>
<tr><td>Now that the degrees are the same, we look at the ratio of the coefficients to determine the horizontal asymptote. What is the ratio of our coefficients?</td><td>$y = \frac{0}{1}$
$y = 0$</td></tr>
<tr><td>Step 2)
This means that we have a horizontal asymptote at $y = 0$. Let's draw and label this on our graph.</td><td>
</td></tr>
<tr><td>Step 3)
Looking at the graph from left to right, we see that it is broken into three segments, separated by the vertical asymptotes. Our table is going to look the same, broken into three groups, separated by the vertical asymptotes. We need to put both vertical asymptotes in the table such that there are three empty spaces on each side. This is the same process as before; this time we just have a larger table, because we have multiple vertical asymptotes.

Remember that the y-values do not exist for our x-values of the vertical asymptotes, so let's put DNE in the y-column for both $x = -3$ and $x = 3$.</td><td>
<table>
<tr><th>x</th><th>y</th></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td>−3</td><td>DNE</td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td>3</td><td>DNE</td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
</table>
</td></tr>
</table>

Teacher	

Let’s continue completing the x-column. Remember, we try to count by ones to keep the points close and symmetric. This gives a better shape to the graph.

x	y
-6	
-5	
-4	
-3	DNE
3	DNE
4	
5	
6	

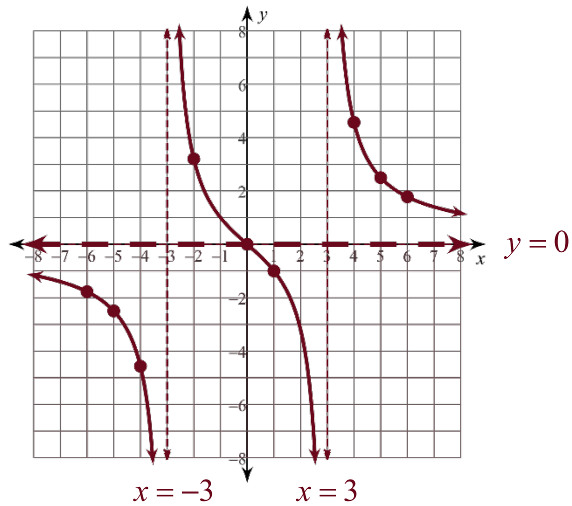
However, we notice that we only need three x-values between our two vertical asymptotes, but there are more than three whole numbers between them.

How do we handle this? You have a couple of options: we could use all of the whole numbers $-2, -1, 0, 1, \& \ 2$. However, since we only need three values, the easier option is to let the middle number be the midpoint, or the number exactly in the middle, of $x=-3$ and $x=3$. So $x=0$ is the middle number in the table. From there, use $-2 \text{ or } -1$ and $1 \text{ or } 2$ on each side of zero.

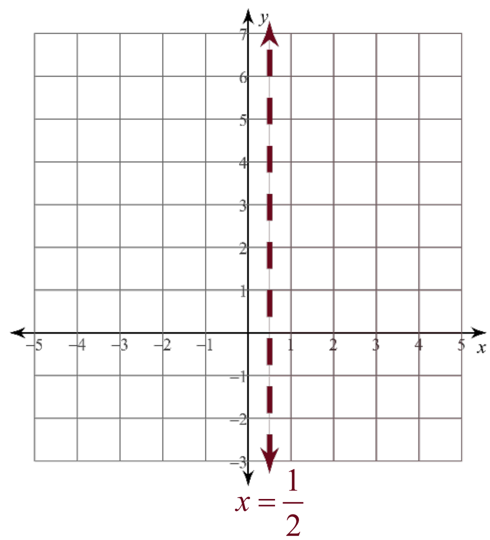
x	y
-6	
-5	
-4	
-3	DNE
-2	
0	
1	
3	DNE
4	
5	
6	

Now we find the y-values, using a scientific calculator to save time.

x	y
-6	-1.8
-5	-2.5
-4	-4.6
-3	DNE
-2	3.2
0	0
1	-1
3	DNE
4	4.6
5	2.5
6	1.8

Teacher	Students
Step 4) Lastly, we plot the points then connect the dots.	
Point out to students that this is an example of a curve crossing the horizontal asymptote and that the ends of the curve are tending towards that horizontal asymptote. Emphasize that vertical asymptotes are not crossed, while horizontal and slant asymptotes can be.	

2) $y = \dfrac{x^2 + 4x}{2x - 1}$

Teacher	Students
Step 1) How do we find the vertical asymptote or asymptotes?	Set the denominator not equal to zero and solve for x. $2x - 1 \neq 0$ $2x \neq 1$ $x \neq \dfrac{1}{2}$
Step 2) If $x \neq \dfrac{1}{2}$, then we have a vertical asymptote at $x = \dfrac{1}{2}$. Let's sketch and label our asymptote.	
Step 1) Now let's find our horizontal asymptote, or at least see if this function has one. How do we find a horizontal asymptote?	Rewrite the equation to make the degrees on the top and bottom of the fraction the same. $y = \dfrac{1x^2 + 4x}{0x^2 + 2x - 1}$
Now that the degrees are the same, we look at the ratio of the coefficients to determine the horizontal asymptote. What is the ratio of our coefficients?	$y = \dfrac{1}{0} = undefined$
This means that we do **not** have a horizontal asymptote. Let's look at the original equation and see if there will be a slant asymptote.	

<table>
<tr><th>Teacher</th><th>Students</th></tr>
<tr><td>The original degree on the top, m, is 2. And the original degree on the bottom, n, is 1. Does $n+1=m$?</td><td>$n+1=m$?
$(1)+1=(2)$?</td></tr>
<tr><td>Yes, so this means that instead of a horizontal asymptote, we have a slant asymptote.

In Algebra 2, we traditionally don’t find the actual equation of the slant asymptote, so we are not going to sketch one. This is something you would do in a later math class, like in precalculus.</td><td></td></tr>
<tr><td>Step 3)
We use the vertical asymptote, $x=\frac{1}{2}$ or $x=0.5$, as the middle number in our table, leaving room for three more x-values on each side.</td><td><table><tr><th>x</th><th>y</th></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr><tr><td>0.5</td><td>DNE</td></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr></table></td></tr>
<tr><td>Let’s continue completing the x-column. Remember, we try to count by ones to keep the points close and symmetric. This gives a better shape to the graph. At the same time, we do not need to over-complicate this problem by just using fractions, so find the nearest whole numbers on each side of one-half, then count by ones from there. We want a nice-looking graph, but not a headache.</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>-2</td><td></td></tr><tr><td>-1</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>0.5</td><td>DNE</td></tr><tr><td>1</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>3</td><td></td></tr></table></td></tr>
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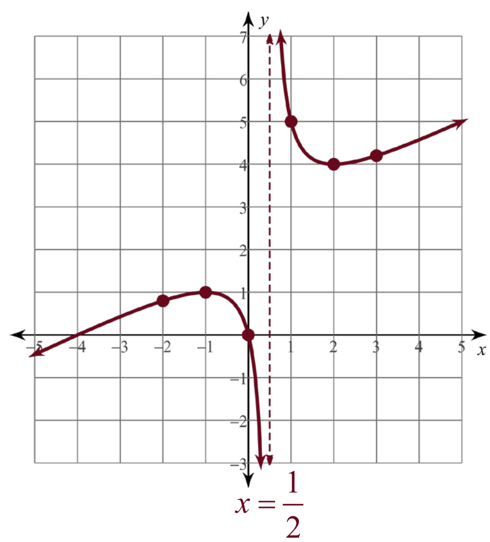
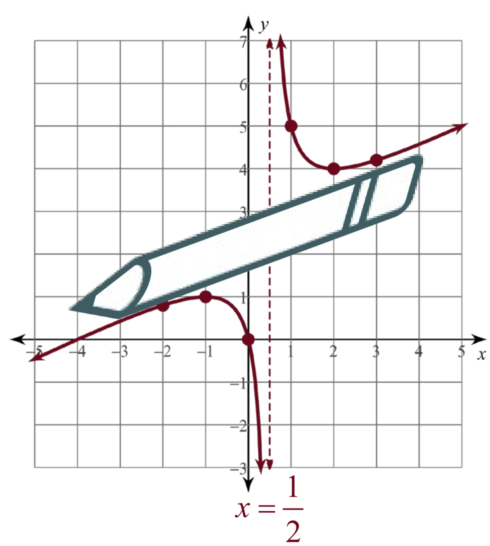
Lastly, we plot the points then connect the dots.</td><td>
</td></tr>
<tr><td>Take your pencil and position it on your graph where you think the slant asymptote would be.

Teacher's note: At this time, you should either circulate the room to check students' papers or model this with a yard stick, etc.

Notice that by having three points on each side of the vertical asymptote, we can get a pretty good idea of where that slant asymptote would be.</td><td>
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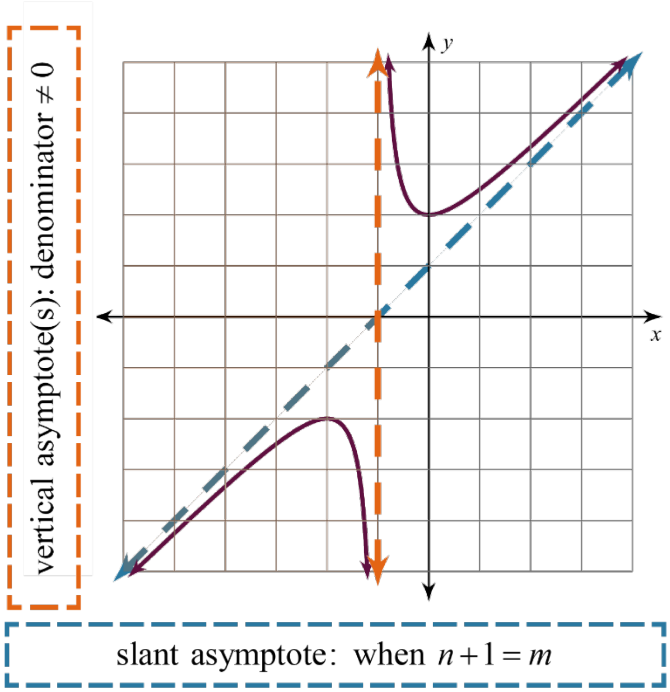
GUIDED NOTES (MODEL NOTES)

Definitions



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Horizontal asymptote: a line with a slope of zero that the curve approaches and **sometimes** crosses*

Slant (oblique) asymptote: a line with a slope other than zero (and is not undefined) that the curve approaches and **sometimes** crosses*

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- Rational functions can have 0 or 1 horizontal asymptotes.
- Rational functions can have 0 or 1 slant asymptotes.

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- If the degree on the top is only 1 greater than the degree on the bottom, then you have a **slant asymptote**.

Step 2) Sketch the asymptote(s) with dashed lines.

- Do not worry about sketching slant asymptotes at this time.

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Examples

Graph each function. Be sure to label the asymptote(s).

1) $y = \dfrac{8x}{x^2 - 9}$

VA: $x = -3$ & $x = 3$

$x^2 - 9 \neq 0$

$x^2 \neq 9$

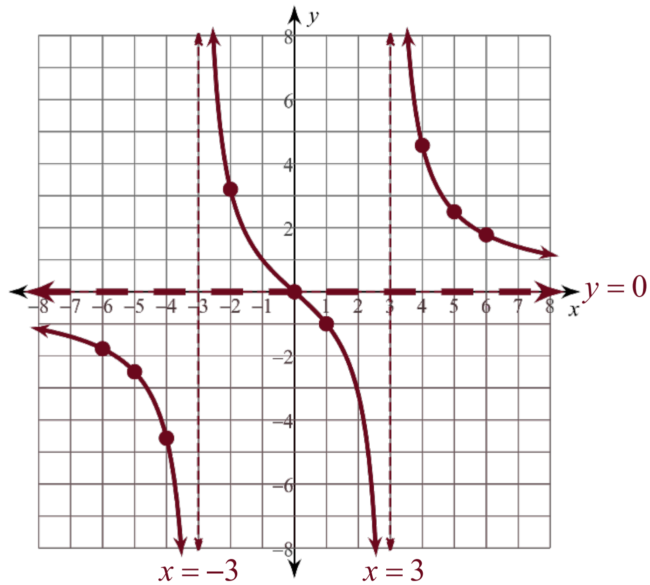
$x \neq \pm 3$

HA: $y = 0$

$y = \dfrac{0x^2 + 8x}{x^2 - 9}$

$y = \dfrac{0}{1} = 0$

x	y
−6	−1.8
−5	−2.5
−4	−4.6
−3	*DNE*
−2	3.2
0	0
1	−1
3	*DNE*
4	4.6
5	2.5
6	1.8



2) $y = \dfrac{x^2 + 4x}{2x - 1}$

VA: $x = \dfrac{1}{2}$

$2x - 1 \neq 0$

$2x \neq 1$

$x \neq \dfrac{1}{2}$

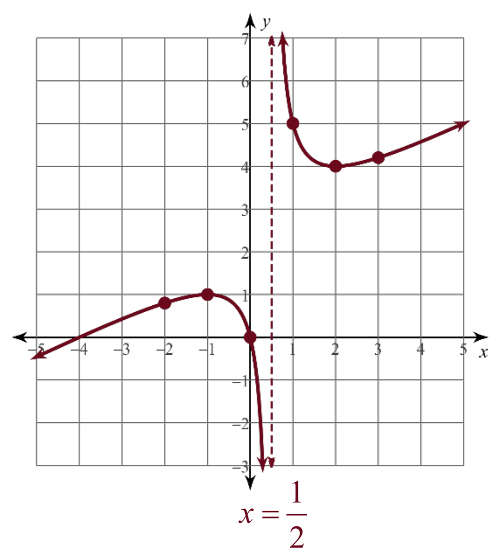
HA: none

$y = \dfrac{x^2 + 4x}{0x^2 + 2x - 1}$

$y = \dfrac{1}{0} = undefined$

$n = 1,\ m = 2 \Rightarrow n + 1 = m \Rightarrow$ slant asymptote

x	y
−2	−0.8
−1	1
0	0
0.5	*DNE*
1	5
2	4
3	4.2



I don't need to draw slant asymptotes, just imagine them.