Take it to the Limit!

Limits Toward Infinity in Rational Functions

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Grade Level	12th Grade	**Time Frame**	2-3 class period(s)
Subject	Mathematics	**Duration**	100 minutes
Course	AP Calculus		

Essential Question

What happens to a rational function at its most extreme values on a graph?

Summary

In this lesson, students will explore the puzzling ideas behind Grandi's series in order to construct an idea of how the sum of an infinite number of terms in a sequence can be evaluated. After exploring Grandi's series, students take a closer look at limits of rational functions as their values of x approach infinity. And lastly, students will extend the idea of a rational functions' convergence/divergence to different types of quotient functions to draw conclusions and generalize how these functions behave at their most extreme x-values.

Snapshot

Engage

Students hypothesize the outcomes to the question, "If you were to add 1-1+1-1+1- . . . forever, what would the result be?"

Explore

Students explore what happens to rational functions as the values of x tend toward infinity.

Explain

Students explain the rules they have uncovered in various rational functions to solidify the concept.

Extend

Students extend their thinking to more difficult types of quotient functions that are not rational by definition.

Evaluate

Students submit their work and reflect on the process.

Standards

Oklahoma Academic Standards for Mathematics (Process Standards)

PK.S: Mathematical Actions and Processes
M.1: Develop a Deep and Flexible Conceptual Understanding: Demonstrate a deep and flexible conceptual understanding of mathematical concepts, operations, and relations while making mathematical and real-world connections. Students will develop an understanding of how and when to apply and use the mathematics they know to solve problems.
M.2: Develop Accurate and Appropriate Procedural Fluency: Learn efficient procedures and algorithms for computations and repeated processes based on a strong sense of numbers. Develop fluency in addition, subtraction, multiplication, and division of numbers and expressions. Students will generate a sophisticated understanding of the development and application of algorithms and procedures.
M.3: Develop Strategies for Problem Solving: Analyze the parts of complex mathematical tasks and identify entry points to begin the search for a solution. Students will select from a variety of problem solving strategies and use corresponding multiple representations (verbal, physical, symbolic, pictorial, graphical, tabular) when appropriate. They will pursue solutions to various tasks from real-world situations and applications that are often interdisciplinary in nature. They will find methods to verify their answers in context and will always question the reasonableness of solutions.
M.4: Develop Mathematical Reasoning: Explore and communicate a variety of reasoning strategies to think through problems. Students will apply their logic to critique the thinking and strategies of others to develop and evaluate mathematical arguments, including making arguments and counterarguments and making connections to other contexts.
M.5: Develop a Productive Mathematical Disposition: Hold the belief that mathematics is sensible, useful and worthwhile. Students will develop the habit of looking for and making use of patterns and mathematical structures. They will persevere and become resilient, effective problem solvers.
M.6: Develop the Ability to Make Conjectures, Model, and Generalize: Make predictions and conjectures and draw conclusions throughout the problem solving process based on patterns and the repeated structures in mathematics. Students will create, identify, and extend patterns as a strategy for solving and making sense of problems.
K.PS2: Develop the Ability to Communicate Mathematically: Students will discuss, write, read, interpret and translate ideas and concepts mathematically. As they progress, students' ability to communicate mathematically will include their increased use of mathematical language and terms and analysis of mathematical definitions.

Attachments

- Limits in Rational Functions Exploration - Spanish.docx
- Limits in Rational Functions Exploration - Spanish.pdf
- Limits in Rational Functions Exploration.docx
- Limits in Rational Functions Exploration.pdf

Materials

- Pencil
- Calculator
- Limits in Rational Functions Exploration handout (attached)

Engage

Have the following question displayed as students enter the classroom, "If you were to add 1-1+1-1+1- . . . forever, what would the result be?"

Ask students to Think-Pair-Share:

1. Have students respond to a question by thinking about it and writing their response down.
2. Ask students to pair with an elbow partner and have each share their responses. They can either choose the “best” response or collaborate together to create a “shared” response.
3. Call on students to share their responses.

Sample Student Responses

Most students will say the answer is either 1 or 0; you may have some students believe the answer is -1 as well. The correct answer is 1/2, but don't give them the answer just yet!

To discuss answers to the question posed, have students form Agreement Circles:

1. Begin by having students form a large circle.
2. Read the statement, "I believe the answer to the question is 1."
3. Ask students to move to the center of the circle if they agree with the statement and stay on the outside if they disagree.
4. When they've found their positions, match the students who agree with the students who disagree proportionally to their answers (e.g., 1:2, 1:3, 1:4, 1:5).
5. Give them a few minutes to defend their ideas in these small groups.
6. Call time, read the statement again, and have students form a large circle again and position themselves according to their opinion after the small group discussions. Their opinions may have changed. Students who agree with the statement move to the inside of the circle.Students who disagree stay on the outside of the circle.
7. Note any changes, and then have students go back to the circle for another round using a new statement.
8. Repeat the process for the following statements: "I believe the answer to the question is 0." "I believe the answer to the question is something besides 1 or 0."

Teacher's Note: Activity Pacing

In the Agreement Circles strategy, you may find that the second and third iteration of the process will go a little quicker than the first. Since all of the proposed questions are similar, the discussion may be short and that's okay.

While students are in agreement circles, pose the question, "What if I told you that the answer to the question is not 1 or 0 but 1/2?" Solicit a few student responses, have students return to their seat, and show the One minus one plus one minus one - Numberphile video.

Embedded video

https://youtube.com/watch?v=PCu_BNNI5x4

Optional Video Alternative

The clip is 11 minutes 9 seconds long. If you would like to demonstrate the process of finding the solution yourself to move the lesson along, this would be the appropriate time to do so. The video does a nice job of using mathematical language and notation for finding limits, which is a great segue to exploring limits as x approaches infinity in rational functions.

After the video (and/or demonstration), field questions from students about the outcomes presented.

Explore

With an Elbow Partner, have students complete the attached **Limits in Rational Functions Exploration** handout.

Explain

After the exploration, use the strategy 4-2-1 to have students agree on the main ideas from the exploration:

1. Ask students to independently generate the four most important ideas on their own.
2. Have students pair up to share their ideas and agree on the two most important ideas from their lists.
3. Now, pair the pairs, making groups of four. Each group must agree on the single most important idea.
4. Ask all students to freewrite individually about the big idea for 3-5 minutes. The goal is to have students explain what they know in such a way that someone who has never heard the idea could understand it.
5. Students return to their small groups of four to participate in a whole-class discussion of the big idea

Sample Student Responses

During the discussion, listen for students to articulate concepts like:

- "If the the higher power of x is in the top, the function will diverge toward either positive or negative infinity."
- "If the highest power of x is in the denominator, then the function will always converge to 0."
- "If the highest powers of x are equal in the top and bottom, then you have to find the quotient of the leading coefficients."

Extend

After the discussion, pose the following limit problems to the groups in the class:

Teacher's Note: Notation Demonstration

Consider writing these on the board using correct notation for students to look at while they try to solve the posed problems.

- Find the limit as x approaches infinity for f(x) = sin(x) / x. Give groups about 3-4 minutes to solve and ask for volunteers to share their answer and their justification for how they arrived at the answer.
- Find the limit as x approaches infinity for f(x) = e^x. Find the limit as x approaches negative infinity for g(x) = e^-x. Give students 3-4 minutes to solve and ask for volunteers to share their answers and their justification for how they arrived at those answers. Follow up with, "How could you write e^-x as a quotient function?"
- Find the limit as x approaches infinity for f(x) = ln(x) / e^x and the limit as x approaches infinity for g(x) = e^x / ln(x). Again, give students 3 - 4 minutes to solve and ask for volunteers to share their answers and their justification for how they arrived at those answers.

Evaluate

Collect student work and ask students to write a Two-Minute Paper to reflect on the day and draw conclusions about the concepts covered.

1. Display the following questions: "What are the three outcomes of limits toward infinity in rational functions, and how can you tell when those outcomes will occur?" and, "What conclusions can you come to regarding limits toward infinity in quotient functions like the problems we solved in groups?"
2. Give participants two minutes to write, then collect their papers.
3. Analyze their responses and share the results with the participants the next day, allowing students the opportunity to hear others' responses.

Resources

- K20 Center. (n.d.). 4-2-1. Strategies. https://learn.k20center.ou.edu/strategy/d9908066f654727934df7bf4f506663d
- K20 Center. (n.d.). Agreement Circles. Strategies. https://learn.k20center.ou.edu/strategy/d9908066f654727934df7bf4f50704ce
- K20 Center. (n.d.). Think-Pair-Share. Strategies. https://learn.k20center.ou.edu/strategy/d9908066f654727934df7bf4f5064b49
- K20 Center. (n.d.). Two-Minute Paper. Strategies. https://learn.k20center.ou.edu/strategy/d9908066f654727934df7bf4f506cf73
- K20 Center. (n.d.). Elbow Partners. Strategies. https://learn.k20center.ou.edu/strategy/116
- One Minus One Plus One Minus One - Numberphile video: Numberphile. (2013, June 25). One minus one plus one minus one - Numberphile [Video file]. Retrieved from https://www.youtube.com/watch?v=PCu_BNNI5x4