Unit Origami: Star-Building on Deltahedra

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Abstract

This workshop provides instructions for folding the star-building unit – a modification of the Sonobe module for unit origami. Geometric questions naturally arise during this process, ranging in difficulty from middle school to graduate levels. Participants will learn to fold and assemble star-building units, then explore the structure of the eight strictly convex deltahedra.

Introduction

Many authors and educators have used origami, the art of paper folding, to provide concrete examples motivating mathematical problem solving. [3, 2] In unit origami, multiple sheets are folded and combined to form a whole; the Sonobe unit is a classic module in this art form. This workshop describes the construction and assembly of the star-building unit[1], highlights a small selection of the many geometric questions motivated by this process, and introduces participants to the strictly convex deltahedra. Proficiency in geometry and origami is not required to enjoy this event.

About the Unit

Three Sonobe units interlock to form a pyramid with an equilateral triangular base as shown in Figure 1. Erecting these pyramids over the faces of structures built from equilateral triangles yields attractive models and provides insight into the composition of such structures [3]. By folding from 6 to 30 star-building units, participants may augment each of the strictly convex deltahedra [1] in turn.

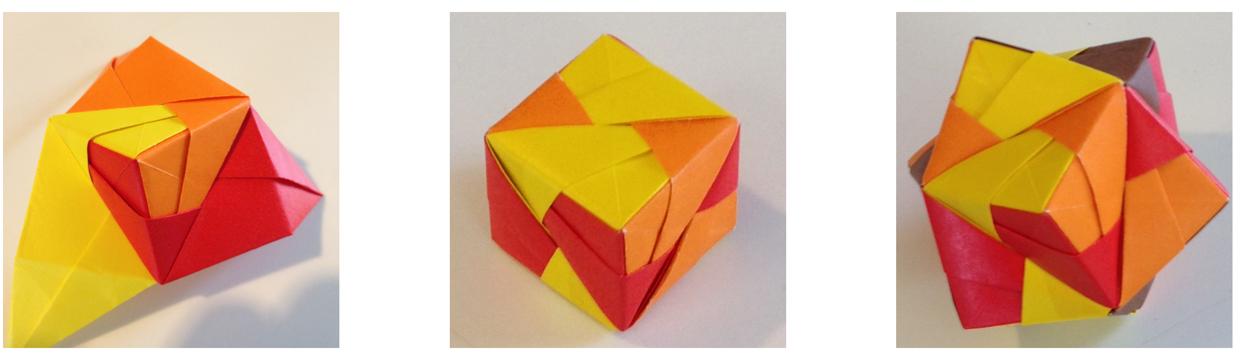

Figure 1 : *Three star-building units interlock to cover an equilateral triangular face.*

[1]Franco's star-building unit is structurally equivalent to the Sonobe unit first published in 1968. [5]

Folding the Unit

To fold the unit, start with a regular 5 inch square of origami paper.[2] Place it colored side down in front of you, and fold it in half vertically. Then fold the right side in to meet the fold you just made.

Exercise: The distance from the left side of the paper to the first fold is $\frac{1}{2}$ the width of the paper. The distance from the left side to the second fold is $\frac{1}{2}+\frac{1}{4}$ paper widths. Imagine opening the right side out again, then folding it in to create a fold $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ paper widths from the left side. A perfect folder with ideal paper could fold forever to get a distance of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ paper widths. What would that look like?

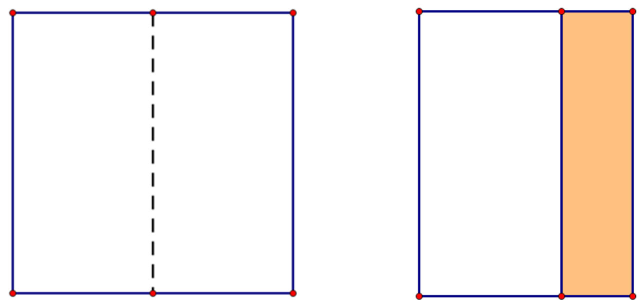

Figure 2 : *Fold vertically, then fold the right side in.*

Fold the left side in to the center, then lay the paper out flat again. You will use the creases just formed as guides for your next four folds.

Fold the upper left corner along the dotted line shown in Figure 3 and crease, so that the top of the paper aligns with the leftmost fold. Then fold the crease you just made down to the leftmost vertical fold. Repeat on the lower right; do not unfold. This folding pattern should be familiar from paper airplane construction.

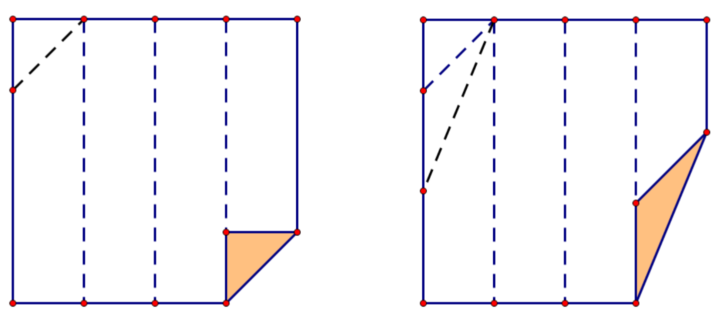

Figure 3 : *Paper airplane folds take upper left and lower right corners to the guide creases.*

Exercise: What are the angles of the shaded triangle on the left side of Figure 3? Side lengths? Area? What about the shaded triangle on the right?

Fold the left and right sides in again to form a rectangle, then fold the top right quarter of the rectangle down to its midline and the lower left quarter up.

Exercise: Can you prove that the figure on the left side of Figure 4 must be a rectangle? That the figure on the right will fold into a parallelogram?

[2] Experienced folders may prefer to use a three inch square for sturdier results, and smaller squares may be used in jewelry making.

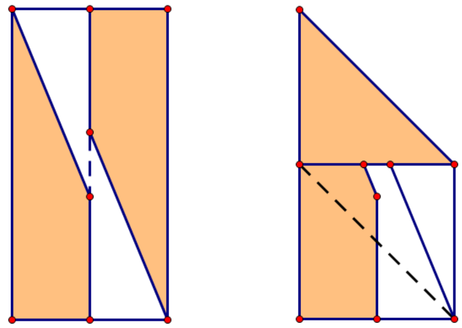

Figure 4 : *Fold into a rectangle, then a parallelogram.*

As shown on the left in Figure 5, take the corners you just folded down and gently tuck them under the paper airplane folds. This creates reinforced points at the upper left and lower right corners of your unit which will be inserted into pockets formed by the paper airplane folds.

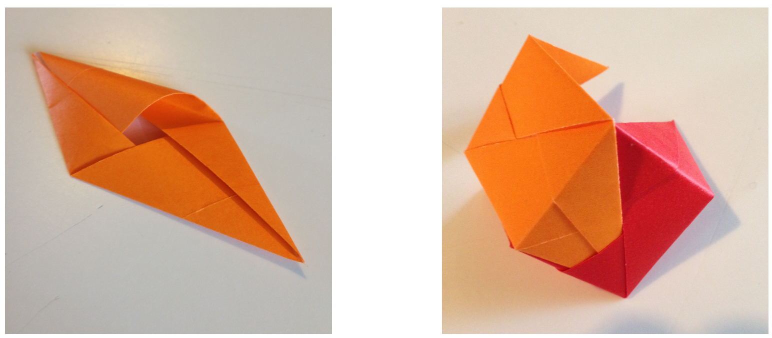

Figure 5 : *Gently tuck corners under paper airplane folds to form pockets and reinforced flaps.*

Flip your unit upside-down; you should see a plain parallelogram with a faint diagonal crease, as shown on the left in Figure 6. Fold the corners of the parallelogram in to make a square. This square will form one face of your cube.

Fold six more units, then assemble them into a cube by inserting reinforced points into central pockets as shown on the right in Figure 5.

Exercise: How are the dimensions of the unit origami cube related to the dimensions of the paper you started with? What size paper would you need to start with to create a cube whose sides have length one centimeter? Whose sides have length one foot?

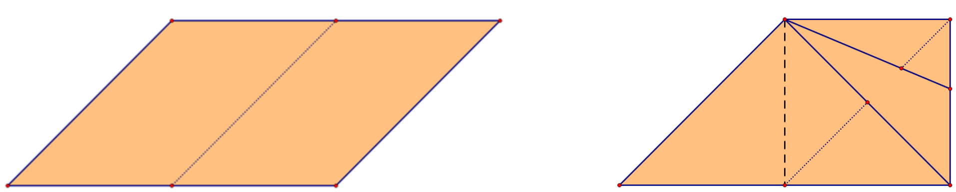

Figure 6 : *Flip, then fold corners in.*

Exercise: Look closely at this cube; in particular, notice the pattern of folds and overlaps on each face. What symmetries does the cube have? What symmetries of the cube preserve the colors of the units that make up the cube—i.e., send each colored unit to one of the same color? How many different symmetry groups can you represent using different colored units? (See Franco [3] for some solutions to this exercise.)

Augmented Convex Deltahedra...

Figure 1 shows three star-building units combined to cover an equilateral triangle; the cube you constructed is built on the skeleton of a tetrahedron or triangular pyramid.

Exercise: Visualize the tetrahedron which was augmented to form your cube.

A natural next step is to use the star-building unit to augment the faces of other structures constructed from equilateral triangles. This class of structures is referred to as *deltahedra*; the strictly convex deltahedra were classified by Rausenberger [6] in 1915 and, independently, by Freundenthal and van der Waerden [4] in 1947. There are exactly 8 such polyhedra: the tetrahedron, triangular bipyramid, octahedron, pentagonal bipyramid, snub disphenoid, triaugmented triangular prism or tetracaidecadeltahedron, gyroelongated square bipyramid, and icosahedron.

Exercise: Use the Euler characteristic

$$(\text{number of vertices}) - (\text{number of edges}) + (\text{number of faces}) = 2$$

and the fact that all faces of a deltahedron are triangles to explain why there are at most 9 strictly convex deltahedra. *Hint:* At most five faces may meet at each vertex.

Exercise: Show that there is no strictly convex deltahedron with 18 faces by attempting to draw the edge graph of such an object.

After creasing the star-building unit along the diagonals of the cube's faces and folding three new units, one has the components to build an augmented triangular bipyramid. Models of the strictly convex deltahedra will be provided at the workshop and their images are readily available online [7]; folding new units three at a time permits a leisurely exploration of the strictly convex deltahedra and their properties.

...and Beyond

As long as the dihedral angles between triangles provide space to erect pyramids above or below the faces, star building units can be assembled to cover any structure made up of equilateral — or nearly equilateral — triangles. Participants are encouraged to explore these structures beyond the workshop, on their own or in collaboration with others.

Exercise (unsolved?): Characterize structures which can be assembled using only star-building units.

References

[1] H. M. Cundy, Deltahedra, *The mathematical gazette*, 36.318 (1952), pp. 263-266.

[2] N. Fout, J. Marker, Folding Your Way to Understanding, *Bridges 2011: Mathematics, Music, Art, Architecture, Culture*, (2011), pp. 671-678.

[3] B. Franco, *Unfolding Mathematics with Unit Origami* (1999), Key Curriculum Press.

[4] H. Freudenthal, B. L. van der Waerden, Over een bewering van Euclides, *Simon Stevin* **25** (1947), pp. 115–128.

[5] D. Lister, Origins of the Sonobe Module, `http://www.britishorigami.info/academic/lister/sonobe.php` (as of April 17, 2015).

[6] O. Rausenberger, Konvexe pseudoreguläre Polyeder, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* **45** (1915), pp. 135-142.

[7] H. Schepker, The Convex Deltahedra, `http://glassgeometry.com/polyhedra/slides/deltahedra.htm` (as of April 17, 2015).